LESSON 5

SOLVING CUBIC EQUATIONS

A cubic expression is an expression of the form $ax^3 + bx^2 + cx + d$. The following are all examples of expressions we will be working with:

$2x^3 - 16,\quad x^3 - 2x^2 - 3x,\ x^3 + 4x^2 - 16,\ 2x^3 + x - 3.$

Remember that some quadratic expressions can be factorised into two linear factors:

e.g. $\underbrace{2x^2 - 3x + 1}_{\text{Quadratic}} = \underbrace{(2x - 1)}_{\text{Linear}}\underbrace{(x - 1)}_{\text{Linear}}$

Now, a cubic expression may be factorised into

(i) a linear factor and a quadratic factor or (ii) three linear factors.

For example, you can easily verify, by multiplying out the right hand side that:

(i) $x^3 - 8 = \underbrace{(x - 2)}_{\text{Linear}}\underbrace{(x^2 + 2x + 4)}_{\text{Quadratic}}$

(ii) $4x^3 - 4x^2 - x + 1 = \underbrace{(x - 1)}_{\text{Linear}}\underbrace{(2x - 1)}_{\text{Linear}}\underbrace{(2x + 1)}_{\text{Linear}}$

There are three types of factorisation methods we will consider:

- Common factor
- Grouping terms
- Factor theorem

Type 1 - Common factor

In this type there would be no constant term.

Example

Example 1

Solve for x: $x^3 + 5x^2 - 14x = 0$

Solution

Solution

$x(x^2 + 5x - 14) = 0$

$\therefore\ x(x + 7)(x - 2) = 0$

$\therefore\ x = 0, x = 2, x = -7$

Type 2 - Grouping terms

With this type, we must have all four terms of the cubic expression. We then pair terms with a common factor and see if a common bracket emerges.

Example

Example 2

Solve for x: $x^3 + 2x^2 - 9x - 18 = 0$

Solution

Solution:

$(x^3 + 2x^2) - (9x + 18) = 0$

$\therefore\ x^2(x + 2) - 9(x + 2) = 0$

$\therefore\ (x+2)(x^2-9)=0$

$\therefore\ (x+2)(x-3)(x+3)=0$

$\therefore\ x=-2, x=3, x=-3$

Type 3 - Using the factor theorem

N.B. If $(x-a)$ is a factor of the cubic expression, then $f(a)=0$.

So, we substitute in values of $x=\pm 1, \pm 2$ … etc until we find a value which makes the expression equal to 0.

Example 3

Example

Solve for x: $x^3-5x+2=0$

Solution

Solution

Try $x=1$: $1^3-5(1)+2=-2$

Try $x=-1$: $(-1)^3-5(-1)+2=6$

Try $x=2$: $2^3-5(2)+2=0$ $\therefore\ (x-2)$ is a factor

$\therefore\ (x-2)(\text{quadratic})=x^3-5x+2$

$(x-2)(x^2+kx-1)=x^3-5x+2$ by inspection.

Compare x terms on LHS and RHS: $-5x=-x-2kx$

$\therefore\ -5=-1-2k$

$\therefore\ k=2$

$\therefore\ x^3-5x+2=(x-2)(x^2+2x-1)=0$

$x=2$ or $x=-1\pm\sqrt{2}$ (using the quadratic formula)

Alternatively, you can use long division to get the factors of x^3-5x+2

Example 4

Example

Solve for x: $2x^3-3x^2-8x-3=0$

Solution

Solution

Try $x=1$: $2(1)^3-3(1)^2-8(1)-3=-12$

Try $x=-1$: $2(-1)^3-3(1)^2-8(-1)-3=0$ $\therefore\ (x+1)$ is a factor

$\therefore\ (x+1)(2x^2+kx-3)=2x^3-3x^2-8x-3$

Compare x^2 terms on both sides:

(N.B. It does not matter whether you compare x^2 or x terms)

$-3x^2=2x^2+kx^2$

$\therefore\ -3=2+k$

$\therefore\ k=-5$

$\therefore\ (x+1)(2x^2-5x-3)=2x^3-3x^2-8x-3=0$

$\therefore\ (x+1)(2x+1)(x-3)=0$

$\therefore\ x=-1, x=-\frac{1}{2}, x=3$

Activity

Activity 1

Solve for x:

1. $2x^3 - x^2 - x = 0$

2. $x^3 - x = 0$

3. $\frac{2}{3}x^3 - 18 = 0$

4. $x^3 + 3x^2 - 4x - 12 = 0$

5. $x^3 - 3x - 2 = 0$

6. $2x^3 + 5x^2 - 14x - 8 = 0$

7. $x^3 + 7x^2 - 36 = 0$

8. $4x^3 + 12x^2 + 9x + 2 = 0$

9. $x^3 - 2x^2 - 4x + 3 = 0$

Activity 2

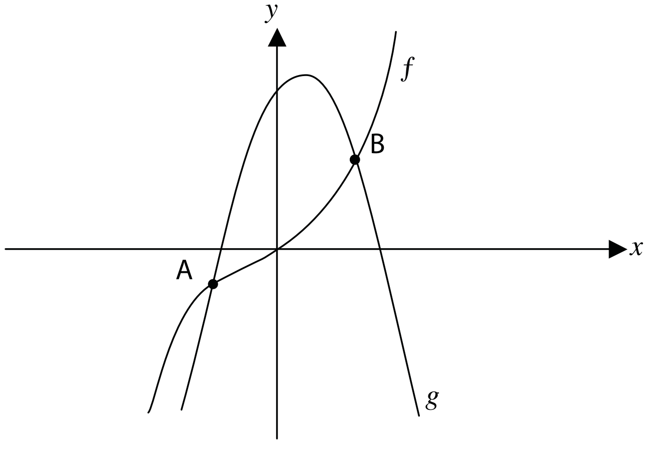
Activity

1. Given that: $f(x) = 6x^3 - 37x^2 + 5x + 6$ and $f(6) = 0$, solve for x, if $f(x) = 0$

2. Solve for x and y if:

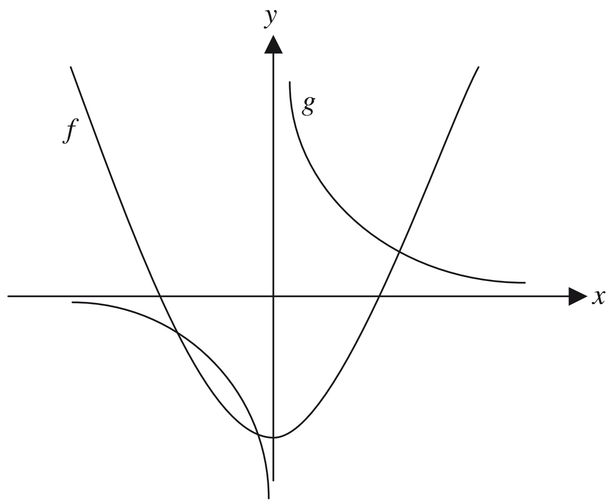
   $y = x^3 + 9x^2 + 26x + 16$ and $y - 3x = 1$

3. In the diagram: $f(x) = x^3$ and $g(x) = -3x^2 + x + 3$



Determine the coordinates of A and B, the points of intersection of *f* and *g*.

4. In the diagram: $f(x) = x^2 - 7$ and $g(x) = \frac{6}{x}$



Make use of the diagram, and a cubic equation, to solve the inequality:
$\frac{6}{x} \geq x^2 - 7$

Solutions to Activities

Activity 1

1. $2x^3 - x^2 - x = 0$

   $\therefore\ x(2x^2 - x - 1) = 0$

   $\therefore\ x(2x + 1)(x - 1) = 0$

   $\therefore\ x = 0 \quad \text{or} \quad x = -\frac{1}{2} \quad \text{or} \quad x = 1$

2. $x^3 - x = 0$

   $\therefore\ x(x^2 - 1) = 0$

   $\therefore\ x(x - 1)(x + 1) = 0$

   $\therefore\ x = 0 \quad \text{or} \quad x = \pm 1$

3. $\frac{2}{3}x^3 - 18 = 0$

$\therefore\ 2x^3 - 54 = 0$

$\therefore\ 2x^3 = 54$

$\therefore\ x^3 = 27$

$\therefore\ x = 7$

4. $x = 2$ is a solution since $2^3 + 3(2)^2 - 4(2) - 12 = 0$

$\therefore\ (x - 2)(x^2 + \mathrm{k}x + 6) = x^3 + 3x^2 - 4x - 12$

Compare x terms on LHS and RHS:

$-2\mathrm{k}x + 6x = -4x$

$\therefore\ -2\mathrm{k} + 6 = -4$

$\therefore\ -2\mathrm{k} = -10$

$\therefore\ \mathrm{k} = 5$

$\therefore\ x3 + 3x2 - 4x - 12 = (x - 2)(x^2 + 5x + 6)$

$\therefore\ (x - 2)(x + 3)(x + 2) = 0$

$\therefore\ x = -3$ or $x = \pm 2$

5. $x = -1$ is a solution since $(-1)^3 - 3(-1) - 2 = 0$

$\therefore\ (x + 1)(x^2 + \mathrm{k}x - 2) = x^3 - 3x - 2$

Compare x terms on LHS and RHS:

$-2x + \mathrm{k}x = -3x$

$\therefore\ -2 + \mathrm{k} = -3$

$\therefore\ \mathrm{k} = -1$

$\therefore\ (x + 1)(x^2 + x - 2) = x^3 - 3x - 2$

$\therefore\ (x + 1)(x - 2)(x + 1) = 0$

$\therefore\ x = -1$ or $x = 2$

6. $x = 2$ is a solution since $2(2)^3 + 5(2)^2 - 14(2) - 8 = 0$

$\therefore\ (x - 2)(2x^2 + \mathrm{k}x + 4) = 2x^3 + 5x^2 - 14x - 8$

Compare x terms on LHS and RHS:

$-2\mathrm{k}x + 4x = -14x$

$\therefore\ -2\mathrm{k} + 4 = -14$

$\therefore\ -2\mathrm{k} = -18$

$\therefore\ \mathrm{k} = 9$

$\therefore\ (x - 2)(2x^2 + 9x + 4) = 2x^3 + 5x^2 - 14x - 8$

$\therefore\ (x - 2)(2x + 1)(x + 4) = 0$

$\therefore\ x = 2$ or $x = -\frac{1}{2}$ or $x = -4$

7. $x = 2$ is a solution since $(2)^3 + 7(2)^2 - 36 = 0$

$\therefore\ (x - 2)(x^2 + kx + 18) = x^3 + 7x^2 - 36$

Compare x terms on LHS and RHS:

$-2kx + 18x = 0x$

$\therefore\ -2k + 18 = 0$

$\therefore\ -2k = -18$

$\therefore\ k = 9$

$\therefore\ (x - 2)(x^2 + 9x + 18) = x^3 + 7x^2 - 36$

$\therefore\ (x - 2)(x + 3)(x + 6) = 0$

$\therefore x = -2$ or $x = -3$ or $x = -6$

8. $x = -2$ is a solution since $4(-2)^3 + 12(-2)^2 + 9(-2) + 2 = 0$

$\therefore\ (x + 2)(4x^2 + kx + 1) = 4x^3 + 12x^2 + 9x + 2$

Compare x terms on LHS and RHS:

$x + 2kx = 9x$

$\therefore\ 1 + 2k = 9$

$\therefore\ 2k = 8$

$\therefore\ k = 4$

$\therefore\ (x + 2)(4x^2 + 4x + 1) = 4x^3 + 12x^2 + 9x + 2$

$\therefore\ (x + 2)(2x + 1)(2x + 1) = 0$

$\therefore\ x = -\frac{1}{2}$ or $x = -2$

9. $x = 3$ is a solution since $(3)^3 + 2(3)^2 - 4(3) + 3 = 0$

$\therefore\ (x - 3)(x^2 + kx - 1) = x^3 + 2x^2 - 4x + 3$

Compare x terms on LHS and RHS:

$-3kx - x = -4x$

$\therefore\ -3k - 1 = -4$

$\therefore\ -3k = -3$

$\therefore\ k = 1$

$\therefore\ (x - 3)(x^2 + x - 1) = x^3 + 2x^2 - 4x + 3$

$\therefore\ x - 3 = 0$ or $x2 + x - 1 = 0$

$\therefore\ x = 3$ $\quad x = \frac{-1 \pm\sqrt{1 - 4(1)(-1)}}{2(1)}$

$x = \frac{-1 \pm\sqrt{5}}{2}$

Activity 2

1. $f(6) = 0$

$\therefore\ (x - 6)$ is a factor of $f(x)$

$\therefore\ f(x) = (x - 6)(6x^2 - x - 1)$

$\therefore\ f(x) = (x - 6)(3x + 1)(2x - 1)$

$\therefore$ If $f(x) = 0$, then

$\therefore\ x = 6$ or $x = -\frac{1}{3}$ or $x = \frac{1}{2}$

2. $x^3 + 9x^2 + 26x + 16 = 3x + 1$

$\therefore\ x^3 + 9x^2 + 23x + 15 = 0$

$x = -1$ is a solution since $(-1)3 + 9(-1)2 + 23(-1) + 15 = 0$

$\therefore\ (x + 1)(x2 + 8x + 15) = 0$

$\therefore\ (x + 1)(x + 5)(x + 3) = 0$

$\therefore\ x = -1$ or $x = -5$ or $x = -3$

3. For co-ordinates of A and B, we have

$\therefore\ x^3 = -3x^2 + x + 3$

$\therefore\ x^3 + 3x^2 - x - 3 = 0$

$\therefore\ x^2(x + 3) - (x + 3) = 0$

$\therefore\ (x + 3)(x^2 - 1) = 0$

$\therefore\ x = \pm 1$ or $x = -3$

4. First, we must find the points of intersection. Therefore:

$x^2 - 7 = \frac{6}{x}$

$\therefore\ x^3 - 7x = 6$

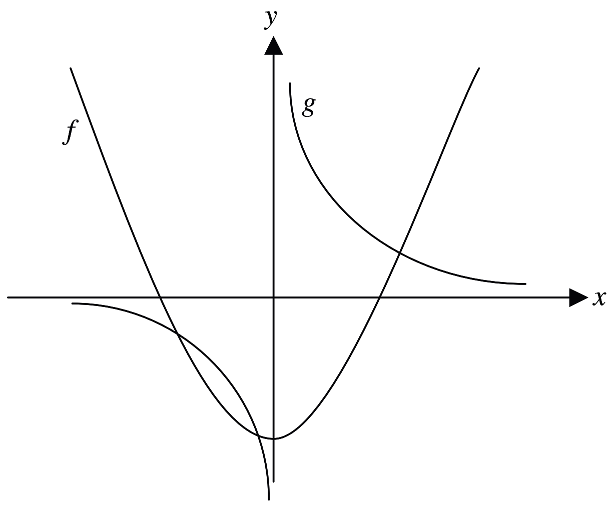
$\therefore\ x^3 - 7x - 6 = 0$

Now, $x = -1$ is a solution since $(-1)^3 - 7(-1) - 6 = 0$

$\therefore\ (x + 1)(x^2 - x - 6) = 0$

$\therefore\ (x + 1)(x + 2)(x - 3) = 0$

$\therefore\ x = -1$ or $x = -2$ or $x = 3$



$\therefore$ Reading solution to $\frac{6}{x} \geq x^2 - 7$ from graph, we get

$0 < x \leq 3$ or $-2 \leq x \leq -1$