CALCULUS 1: LIMITS, AVERAGE GRADIENT AND FIRST PRINCIPLES DERIVATIVES

Learning Outcome 2: Functions and Algebra
**Assessment Standard 12.2.7 (a)**

In this section:

- The limit concept and solving for limits
- Function notation (brief revision section)
- Finding the rate of change of a function between two points or average gradient
- Calculating the derivative of a function from first principles

Limit Concept

Background

The concept of finding a limit looks at what happens to a function value (*y*-value) of a curve, as we get close to a specific *x*-value on the curve. One way to think of it is: if you start walking in a certain direction and find yourself heading towards a big boulder, this boulder will be your limit.
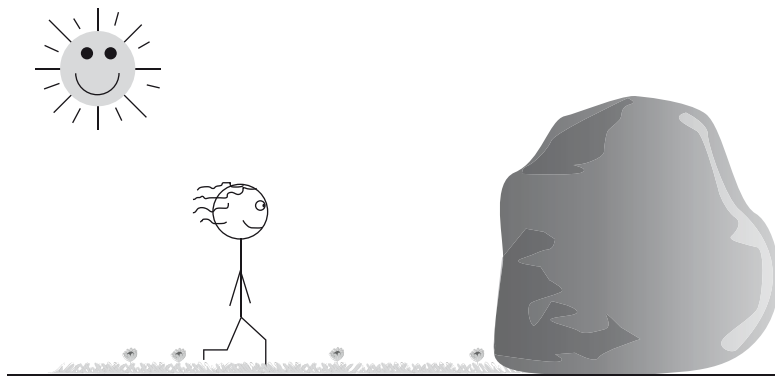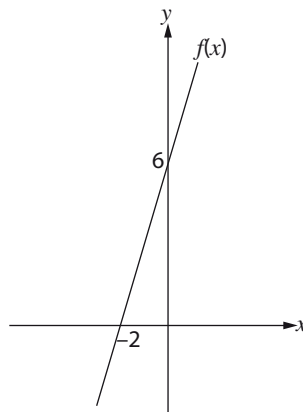
Example

Example 1

Let's start by looking at the function $f(x) = 3x + 6$ shown below:

Now let's consider the $\lim_{x \to 0} 3x + 6$ (we say: the limit of $f(x)$ as x tends to zero). So, we want to know what the *y*-value is equal to when approaching $x = 0$.
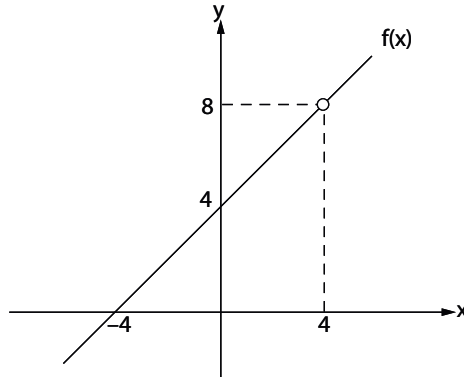
We say:

$$\lim_{x \to 0} f(x) = \lim_{x \to 0} 3x + 6$$
$$= 3(0) + 6 \qquad \text{Substitute in } x = 0, \text{ and remove } \lim_{x \to 0}$$
$$= 6$$

Example 2

In the case of $f(x) = \frac{x^2 - 16}{x - 4}$ (shown below) we can use a limit to find what value $f(x)$ tends to as x tends to four.

But if we substitute $x = 4$ into the limit, we will get $\lim_{x \to 4} f(x) = \frac{0}{0}$, but we know that this is wrong.

So how can we solve this algebraically?

Well, let's see if we can simplify $f(x)$ at all:

$f(x) = \frac{x^2 - 16}{x - 4}.$ The numerator is a difference of two squares, so we can factorise

$= \frac{(x - 4)(x + 4)}{x - 4}$ The $(x - 4)$ in the numerator and denominator will cancel

$= x + 4$

Now let's solve for the limit of $f(x)$:

$$\lim_{x \to 4} f(x) = \lim_{x \to 4} \frac{x^2 - 16}{x - 4}$$
$$= \lim_{x \to 4} x + 4 \qquad \text{Now we can substitute in for } x = 4$$
$$= 8 \qquad\qquad \text{this means that } f(x) \text{ tends to 8 but does not equal 8 at the } x \text{ value of 4}$$

Note that if substituting for the limit produces a zero denominator, factorise and cancel first!

Activity 1

1) $\lim_{x \to 5} x^2 + 2$

2) $\lim_{x \to -1} \frac{x^2 + 2x + 1}{x + 1}$

3) $\lim_{x \to 1} \frac{1}{2x + 2} + \frac{1}{x}$

Revision of Function Notation

Example **Example 1**

If $f(x) = x^2 - 3x + 2$, find the value of:

1. $f(2)$

    $f(2)$ tells us that in the formula $f(x)$, we must make the substitution $x = 2$

    $f(2) = (2)^2 - 3(2) + 2$

    $= 4 - 6 + 2$

    $= 0$ this tells us that on the curve $f(x)$, at the point $x = 2$, $f(2) = 0$.

    ∴ In actual fact we have found the co-ordinates (2; 0) of a point on the curve $f(x)$.

2. $f(-1)$

    $f(-1)$ tells us to make the substitution $x = -1$ in $f(x)$

    $f(-1) = (-1)^2 - 3(-1) + 2$

    $= 1 + 3 + 2$

    $= 6$

3. $f(x + 2)$

    $f(x + 2)$ we substitute $x = x + 2$ into $f(x)$

    $f(x + 2) = (x + 2)^2 - 3(x + 2) + 2$ multiply out

    $= x^2 + 4x + 4 - 3x - 6 + 2$ simplify

    $= x^2 + x$

4. $f(x + h)$ in this case we must substitute $x = x + h$ into $f(x)$.

    $f(x + h) = (x + h)^2 - 3(x + h) + 2$ multiply out

    $= x^2 + 2xh + h^2 - 3x - 3h + 2$ this cannot be simplified further and is therefore our final answer

Example **Example 2**

If $f(x) = x^2 + x$, determine $\dfrac{f(x + h) - f(x)}{h}$:

$\dfrac{f(x + h) - f(x)}{h}$

$= \dfrac{(x + h)^2 + (x + h) - (x^2 + x)}{h}$ Substitute $x = x + h$ for $f(x + h)$

$= \dfrac{x^2 + 2xh + h^2 + x + h - x^2 - x}{h}$ Remember to distribute the negative sign

$= \dfrac{h(2x + h + 1)}{h}$ Factorise and take a common factor of h

$= 2x + h + 1$ This cannot be simplified further and is our final answer

Average Gradient

Background

When dealing with straight line graphs, it is easy to determine the slope of the same graph by calculating the change in x-value between two points and the change in y-value between the same two points. Then we substitute these into the formula for gradient $= \frac{\text{change in } y}{\text{change in } x}$. However, when dealing with curves the gradient changes at every point on the curve, therefore we need to work with the average gradient. The average gradient between two points is the gradient of a straight line drawn between the two points.

Example

Given the equation $f(x) = 2x^2 - x - 6$, find the average gradient between $x = 1$ and $x = 4$.

First we need to find the value of $f(x)$ at $x = 1$. This is the y-value of $f(x)$ at $x = 1$.

$f(1) = 2(1)^2 - 1 - 6$ (making the substitution $x = 1$)

    $= 2 - 7$

    $= -5$

Next we must find the value of $f(x)$ at $x = 4$

$f(4) = 2(4)^2 - 4 - 6$

    $= 2(16) - 10$

    $= 22$

Now we can substitute these values into the average gradient formula:

$$\frac{y_B - y_A}{x_B - x_A} = \frac{22 - (-5)}{4 - 1}$$
$$= \frac{27}{3} = 9$$

The Derivative (First Principles)

Background

We will now investigate the average gradient between 2 points. By calculation we can show that this is equivalent to:
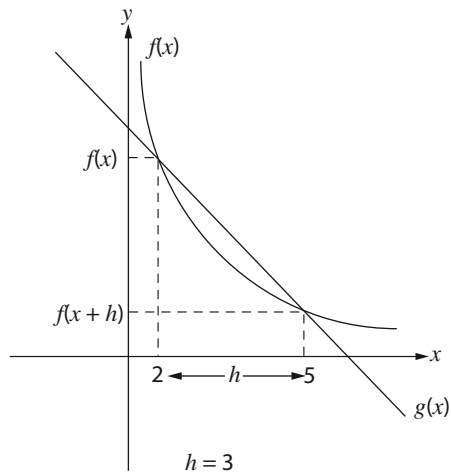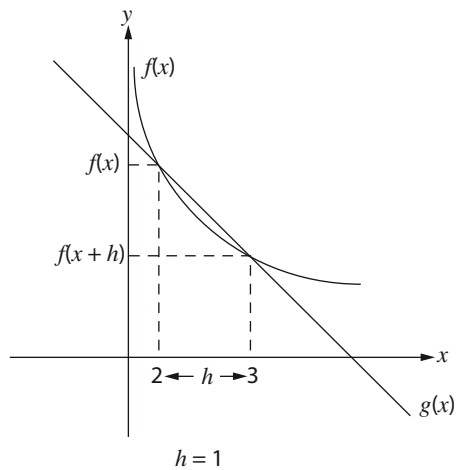
$$\frac{f(x + h) - f(x)}{h}$$
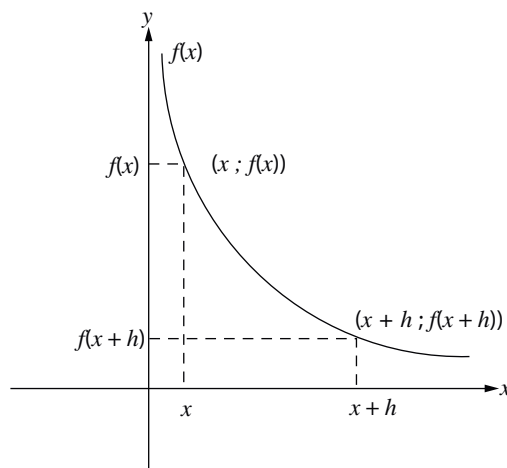
Now think about it: as h, the distance between your 2 x-values, gets smaller your 2 points get closer together. Lets look at an example of this: let $x = 2$ for the graph below. ($\therefore h = 3$, since $5 - 2 = 3$)
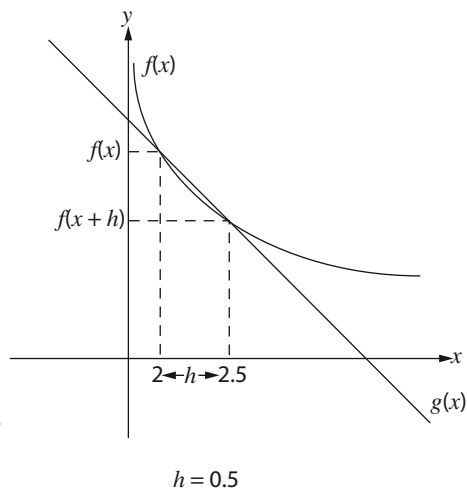
for $h = 3$:

$h = 3$

for $h = 1$:

$h = 1$

for $h = 0.5$

$h = 0.5$

In each of the graphs above, $g(x)$ is a straight line with the same gradient as the average gradient of the curve between the points x and $x + h$.

Can you see that as h decreases the two points on the x axis get closer together?

Can you also see that as the two poins on the x-axis get closer together, the average gradient changes?

So as the two points get closer together (as h becomes 0) the average gradient between the two points becomes the actual gradient at the point. However, h

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cannot equal 0 as that would give us a 0 denominator, so we must use limits. The actual gradient at a point is called the derivative.

To find the limit of the gradient as h tends to zero, we use the formula: $\lim\limits_{h \to 0} \dfrac{f(x + h) - f(x)}{h}$. We call this the derivative of the function and use the notation $f'(x)$ This method is called finding the derivative from first principles.

Example

Find the derivative of the function $f(x) = 3x^2$ from first principles.

Solution:

$f(x + h) = 3(x + h)^2$

$\qquad = 3(x^2 + 2xh + h^2)$

| Start by finding $f(x + h)$, which is the first part of the 'formula' |

$\qquad = 3x^2 + 6xh + 3h^2$

Now substitute the expressions for $f(x)$ and $f(x + h)$ into the first principles formula:

$f'(x) = \lim\limits_{h \to 0} \dfrac{f(x + h) - f(x)}{h}$

| Dont forget to change the signs of the term in the second bracket |

$\qquad = \lim\limits_{h \to 0} \dfrac{(3x^2 + 6xh + 3h^2) - (3x^2)}{h}$

$\qquad = \lim\limits_{h \to 0} \dfrac{3x^2 + 6xh + 3h^2 - 3x^2}{h}$ This can be simplified further

$\qquad = \lim\limits_{h \to 0} \dfrac{6xh + 3h^2}{h}$

| If we were to evaluate this now, we would get $f'(x) = \frac{0}{0}$. But we have a common factor of h in the numerator, so let's factorise. |

$f'(x) = \lim\limits_{h \to 0} \dfrac{h(6x + 3h)}{h}$ The h in the numerator will cancel out the h in the denominator

$\qquad = \lim\limits_{h \to 0} (6x + 3h)$

$\qquad = 6x + 3(0)$ Substitute in for $h = 0$, and remove $\lim\limits_{h \to 0}$

$\qquad = 6x$ This is the derivative of $f(x) = 3x^2$

Activity 2

1.   For the curve $f(x) = -x^2 + 9$, find:

1.1   the average gradient between $x = 0$ and $x = 6$

1.2   the derivative, from first principles, at $x = 3$

2.   From first principle, find $f'(x)$ for the following functions:

2.1 $f(x) = x^2 + 2$

2.2 $f(x) = 3x^2 + 3x$

Calculus 2: General Rules for Differentiation, and Tangents

Learning Outcome 2: Functions and Algebra
Assessment Standard 12.2.7 (a)

In this section we will:

- determine the general rules for differentiation
- determine the gradient and equation of a tangent to a function at a desired point

General Rules for Differentiation

Note: there are many different ways of writing $f'(x)$, we can use:

- $D_x[f(x)]$ we say: the derivative of $f(x)$ with respect to x
- $\dfrac{dy}{dx}$ we say: the derivative of y with respect to x
- y' or f' we say: the derivative of y or $f(x)$

Derivative Laws

I am sure you will agree that finding the derivative from first principles is a rather time-consuming process and that a more time-efficient method is needed. Luckily for you, there is one! A set of rules have been created in order to speed up the process, and these are:

1.   If $f(x) = kx^n$ where k is a constant, then $f'(x) = k \cdot nx^{n-1}$

- If $f(x) = 6x^3$, then $f(x) = 6 \cdot 3x^2 = 18x^2$

2.   If $f(x) = k$, where k is a constant, then $f'(x) = 0$

If the $f(x)$ is equal to a constant, then the derivative of the function is zero.

- If $f(x) = 3$, then $f'(x) = 0$

3.   If $f(x) = g(x) + h(x)$, then $f'(x) = g'(x) + h'(x)$

The derivative of a sum is the sum of the derivatives

- If $f(x) = 6x^3 + x^6$, then $f'(x) = 18x^2 + 6x^5$

4.   Similarly, if $f(x) = g(x) - h(x)$, then $f'(x) = g'(x) - h'(x)$

The derivative of a difference, is the difference of the derivatives

- If $f(x) = 6x^3 - x^6$, then $f'(x) = 18x^2 - 6x^5$

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Remember:

- $\frac{1}{x^2} = x^{-2}$ If you have x in the denominator, take it to the numerator and change the sign of the exponent
- $\sqrt[3]{x} = x^{\frac{1}{3}}$ If you have a surd, change it to power form.
- $\frac{4x^2 - x + 2}{x} = 4x - 1 + \frac{2}{x}$ If you have many terms over a common denominator, with a single term in the denominator, separate the terms out into several fractions

Examples:

Find the derivatives of the following functions

1.1 $f(x) = 3x^2 + 6x$

$\therefore f'(x) = 6x + 6$

1.2 $f(x) = 3x^3 + \frac{1}{2}x + 4$

$\therefore f'(x) = 9x^2 + \frac{1}{2}$

1.3 $f(x) = \frac{3}{x^2}$

$= 3x^{-2}$ Take the 'x' to the numerator by changing the sign of the exponent

$f'(x) = 3(-2x^{-3})$

$= -6x^{-3}$

$= -\frac{6}{x^3}$

1.4 $f(x) = 2x^2 - \frac{2}{3x^3} + 7$

$= 2x^2 - \frac{2}{3}x^{-3} + 7$

$f'(x) = 4x + 2x^{-4} + 0 = 4x + \frac{2}{x^4}$

Activity 3

Activity

Find the derivatives of the following functions:
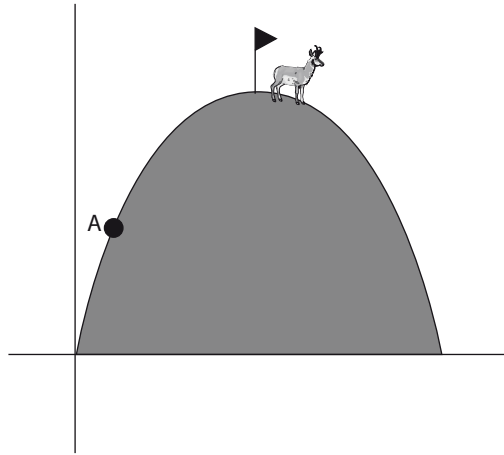
1. $f(x) = \sqrt{x} + \frac{1}{\sqrt{x}}$

2. $y = 2x^3(2x - 3)^2$

3. $y = \frac{x^2 - 2x + 2}{x}$

Derivatives are useful in many aspects of life, one example is:

- If a mountaineer was deciding which mountain he is capable of climbing, he would want to know two things about the mountain: how high it is and how steep it is. If we consider the mountain as a parabola, it would be represented as seen above. Then the height of the mountain at point A could be determined by substitution of the x-value of A into $f(x)$. The steepness of the slope of the mountain could be found by taking the derivative of the parabola and evaluating it at point A. This is eqiuvalent to finding the gradient of the tangent at A.

Finding the Equation of the Tangent

Background

The tangent to a point on the curve is a straight line that touches the curve at that single point and has the exact same gradient as the curve at that point.

To find the gradient of the tangent, we need to find the gradient of the curve at that exact point. To do this, we find the derivative of the function at that point (by substituting the x-value into the derivative) and this equals the gradient.
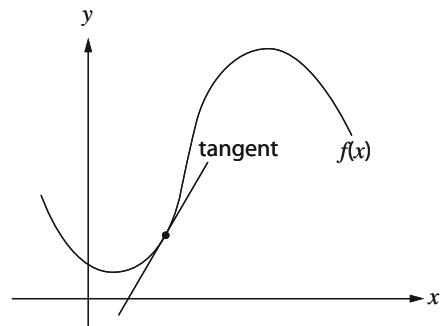
Remember a tangent is a straight line and so has equation $y = mx + c$. We already have m (the gradient) as shown above and to get c, we simply substitute the coordinates of the point of contact between tangent and curve into the equation $y = mx + c$. So that's how we find the equation of a tangent to a curve using the derivative.

Example

Example 1

If we look at the function $f(x) = x^2 + 2x + 1$ and want to find the gradient of the tangent at the point $x = 4$, first we need to find the derivative of $f(x)$:

$f'(x) = 2x + 2$

This is the general formula for the gradients of all tangents to the curve $f(x)$, called the gradient function, but we want to know the specific value of the gradient at $x = 4$, so we must substitute $x = 4$ into the derivative:

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$$m_{\text{Tangent}} = 2(4) + 2$$
$$= 10$$

Therefore, the gradient of the tangent to $f(x)$ at the point $x = 4$ is 10.

If we want to find the equation of the tangent, we need a point on the curve. Since we only know the x-value in this case, we need to find the y-value at $x = 4$, so we use the original curve to get this y-value

$f(4) = (4)^2 + 2(4) + 1$ making the substitution $x = 4$ into $f(x)$

   $= 16 + 8 + 1$

   $= 25$ so we have the point on $f(x)$: (4, 25)

We know that the general formula for a straight line graph is $y = mx + c$. Now that we have found a point on the graph and we know the gradient, we can solve for c:

$25 = 10(4) + c$ substituting in $y = 25$, $x = 4$ and $m = 10$

$25 = 40 + c$

$25 - 40 = c$

$c = -15$

Therefore, the equation for the tangent to $f(x)$ at $x = 4$ is $y = 10x - 15$

Example 2

Example

Consider $f(x) = -x^2 - 2x + 8$

1.1   Sketch: $f(x)$

1.2   Determine: $f'(x)$

1.3   Determine: $f'(1)$

1.4   What does this number represent?

1.5   Hence determine the equation of the tangent to $f(x)$ at $x = 1$

1.6   Determine the equation of the tangent at $x = -2$.

Solution

Solution

1.1   Sketch: $f(x)$

First find intercepts with the axes:

Let $x = 0$ to find the y-intercept, so $y = 8$ and the y-intercept is at (0, 8)

To find the x-intercepts let $y = 0$, so $0 = (-x + 2)(x + 4)$ and the x-intercepts are at (2 ; 0) and (−4 ; 0)

Axis of symmetry: $x = \dfrac{-b}{2a}$

                      $= \dfrac{-(-2)}{2(-1)}$

                      $= \dfrac{2}{-2}$

                      $= -1$

For the turning point we must find the y-value at the axis of symmetry:
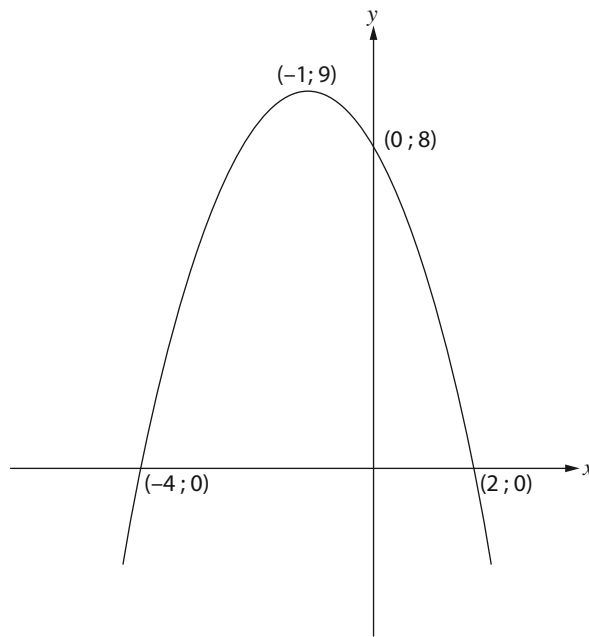
$y = -(-1)^2 - 2(-1) + 8$

  $= -1 + 2 + 8$

  $= 9$

- Turning Point $(-1; 9)$

Now we can sketch $f(x)$:

1.2 Determine: $f'(x)$

$f'(x) = -2x - 2$

1.3 Determine: $f'(1)$

$f'(1) = -2(1) - 2$ substitute $x = 1$ into $f'(x)$

$= -4$

1.4 What does this number represent?

The gradient of the tangent at $x = 1$

1.5 Hence determine the equation of the tangent to $f(x)$ at $x = 1$

We have already found the gradient of the tangent at $x = 1$. Now we need to find a point on $f(x)$ and solve for the c from $y = mx + c$. We need to find the y-value of the curve at $x = 1$, so we need to find $f(1)$:

$f(1) = -(1)^2 - 2(1) + 8$

$= -1 - 2 + 8$

$= 5$

$\therefore$ $(1; 5)$ is the point on $f(x)$ at which we are trying to find the tangent

Now we can use this to solve for c:

$5 = (-4)(1) + c$ substitute in $y = 5$, $m = -4$ and $x = 1$

$5 = -4 + c$

$c = 4 + 5$

$\therefore c = 9$

$\therefore y = -4x + 9$ is the tangent to $f(x)$ at $x = 1$

1.6   Determine the equation of the tangent at $x = -2$

First we need to find $f'(-2)$

$f'(-2) = -2(-2) - 2$

$= 4 - 2$

$= 2$ This is the gradient of the tangent of $f(x)$ at $x = -2$

Now we need to find the corresponding y-value at this point

$f(-2) = -(-2)^2 - 2(-2) + 8$

$= -4 + 4 + 8$

$= 8$

$\therefore (-2; 8)$ is the point on $f(x)$ at which we are trying to find the tangent

We have everything we need to solve for c in the equation (substitute the point $(-2;8)$)

$y = mx + c$:

$y = 2x + c$

$8 = -4 + c$

$c = 12$

$\therefore y = 2x + 12$ is the tangent to $f(x)$ at $x = -2$

Activity 4   Activity

1.   Find the equation of the tangent to the curve $f(x) = (x - 1)^2$ at the point where the graph meets the y-axis

2.   Find the equation of the tangent to the curve $f(x) = -x^2 + x + 2$ at the point $(1 ; 2)$

3.   The line $y = 11x + 44$ is a tangent to the curve of $f(x) = -2x^2 - 5x + 12$. Determine the point of contact.

4. Find the equation of the tangent to the curve of $f(x) = \frac{3}{x}$ at the point $x = -1$

Calculus 3: Sketching and Analysing Graphs Using Calculus

Learning Outcome 2: Functions and Algebra
Assessment Standard 12.2.7 (a)
In this section:

- Sketching a cubic graph
- Finding the equation of a cubic graph
- How to work with points of Inflections
- Application of maxima and minima in real-life scenarios
- Calculating the rate of change in real-life scenarios

Cubic Graphs

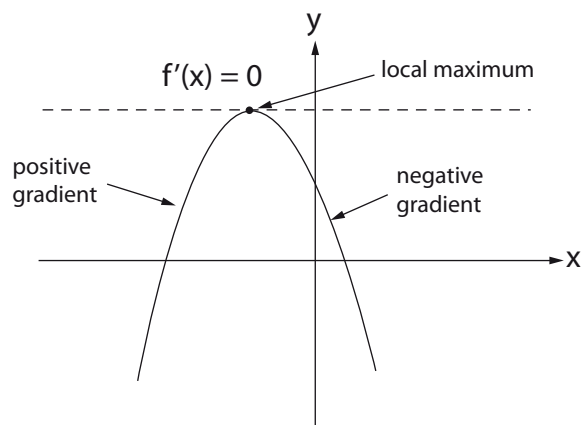### Background

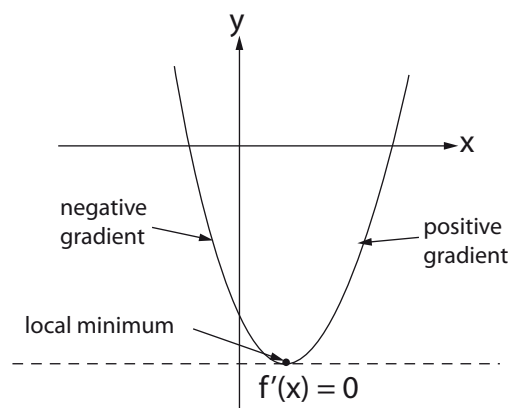The cubic graph has a general formula: $f(x) = ax^3 + bx^2 + cx + d$.

Let's first look at some new concepts and terminology that must be learnt:

- **Stationary point:** a point on the function where the gradient is zero. There are three kinds of stationary points:
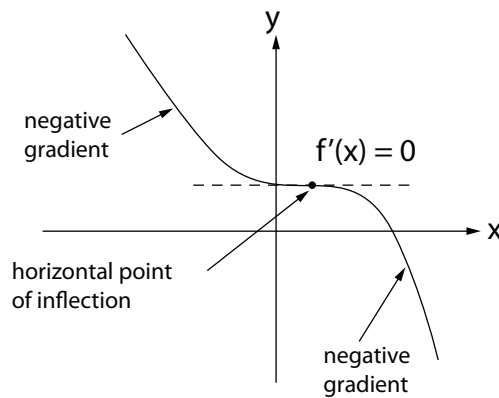
1. Local maximum:

2. Local Minimum:
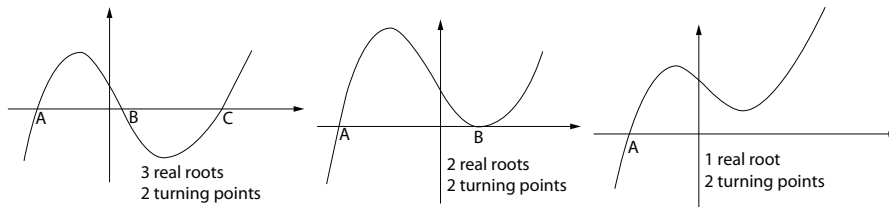
3.   Point of Inflection:

BE CAREFUL: Not all points of inflection are stationary points! The gradient of the curve MUST BE ZERO at a stationary point.
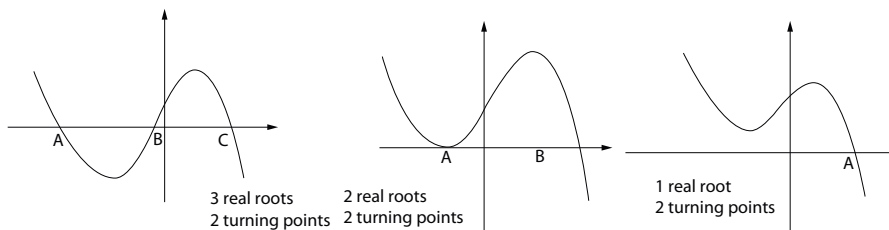
As with curve sketching in the past:

- For y-intercept, put $x = 0$ then solve for y, i.e. $y = d$. Intercept at $(0; d)$
- For x-intercepts, put $y = 0$, i.e. solve $ax^3 + bx^2 + cx + d = 0$. There could be one, two or three real roots.
- For stationary points, put $f'(x) = 0$ and solve for x, i.e. $3ax^2 + 2bx + c = 0$, then use these values to find the y-value of the curve at these points.
- Find the points of inflection by setting the second derivative equal to zero. Points of inflection, $f''(x) = 0$

There could be:

- TWO stationary points, in which case the possible graphs for a>0 are:

3 real roots
2 turning points

2 real roots
2 turning points

1 real root
2 turning points

And for $a < 0$:

3 real roots
2 turning points

2 real roots
2 turning points

1 real root
2 turning points

- ONE stationary point, in which case the possible graphs are:

Point of inflection
1 real root

Point of inflection
1 real root
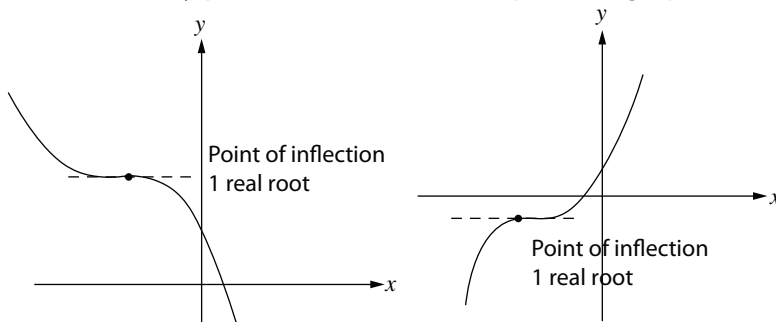
- NO stationary points: in this case you may have a point of inflection for which the gradient is not zero.
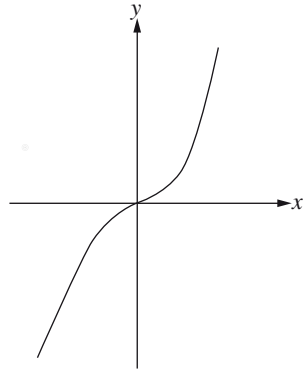
Example 1 (Curve Sketching)

$f(x) = x^3 - x^2 - x + 10$

Sketch $f(x)$, clearly showing all intercepts with the axes and stationary points.

Stationary Points: A good way to start is to find the stationary points. In order to do this, we must find $f'(x) = 0$ since at the minimums and maximums, the gradient of the curve is 0.

$f'(x) = 3x^2 - 2x - 1$

$3x^2 - 2x - 1 = 0$ factorise

$(3x + 1)(x - 1) = 0$

$x = -\frac{1}{3}$ OR $x = 1$

Now we need to find the y-values of the curve at the stationary points, so we must find $f\left(-\frac{1}{3}\right)$ and $f(1)$.

Substitute $x = -\frac{1}{3}$ into $f(x)$

$$f\left(-\frac{1}{3}\right) = \left(-\frac{1}{3}\right)^3 - \left(-\frac{1}{3}\right)^2 - \left(-\frac{1}{3}\right) + 10$$
$$= -\frac{1}{27} - \frac{1}{9} + \frac{1}{3} + 10$$
$$= \frac{-1-3+9+270}{27}$$
$$= \frac{275}{27}$$
$$= 10\frac{5}{27} \approx 10{,}2$$
$$\left(-\frac{1}{3}; 10\frac{5}{27}\right)$$

Substitute $x = 1$ into $f(x)$

$$f(1) = 1 - 1 - 1 + 10$$
$$= 9$$
$$(1; 9)$$

These are the two points on $f(x)$ at which we have stationary points.

From the stationary points, we can now make a rough sketch of the graph:

Points of Inflection: to find if there are any points of inflection, we need to solve for $f''(x) = 0$

$$f''(x) = 6x - 2$$
$$6x - 2 = 0$$
$$6x = 2$$
$$x = \frac{1}{3}$$

∴ there is a point of inflection at $x = \frac{1}{3}$. We need to find $f(\frac{1}{3})$
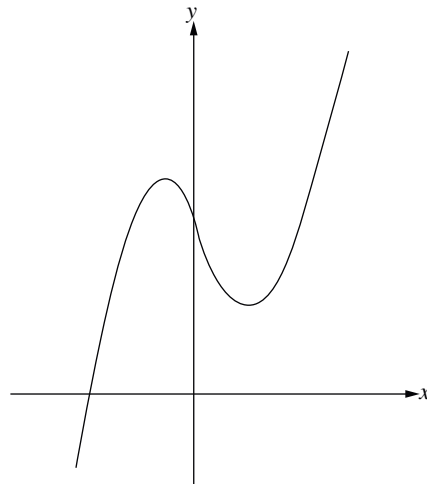
$$f(\tfrac{1}{3}) = (\tfrac{1}{3})^3 - (\tfrac{1}{3})^2 - (\tfrac{1}{3}) + 10$$
$$= \tfrac{1}{27} - \tfrac{1}{9} - \tfrac{1}{3} + 10$$
$$= \tfrac{1 - 3 - 9 + 270}{27}$$
$$= \tfrac{259}{27}$$
$$= \left(\tfrac{1}{3} \, ; \, \tfrac{259}{27}\right)$$

x-intercepts: Next we must find the x-intercepts by making $f(x) = 0$

Cubic functions are more difficult to factorize than quadratic functions. Using the factor theorem is often the easiest way.

REMEMBER: when using the factor theorem, if we find $f(a) = 0$ for some value, a, then $(x - a)$ is a factor of $f(x)$.

Use trial and error to get your first factor.

Try $f(-1)$ $f(1)$, $f(-2)$, and $f(2)$ until you get one equal to zero…

$f(1)$; $f(-1) \neq 0$; $f(2) \neq 0$ but

$f(-2) = -8 - 4 + 2 + 10 = 0$

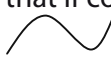∴ $x + 2$ is a factor. We usually get the other factor(s) by doing long division, synthetic division or the 'inspection method. By looking at the rough sketch above can you see why there is only 1 x-intercept
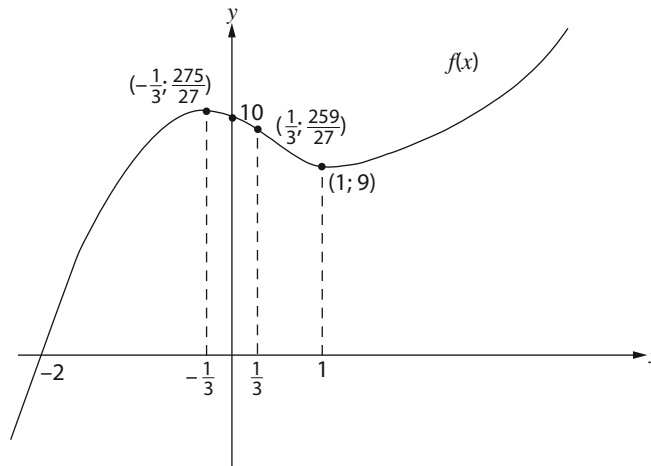
y-intercept: to find the y-intercept, let $x = 0$

$y = (0)^3 - (0)^2 - 0 + 10$

$= 10$

Now we can sketch the graph by joining all the points: note that if coefficient of x^3 in the original equation is positive, the graph has shape /‿\/ and if the coefficient of x^3 is negative, \‿/\.

In this case we have $1x^3$ so the shape is as follows.

Example 2 (Finding the Equation of a Curve)

Find the equation of $f(x)$:
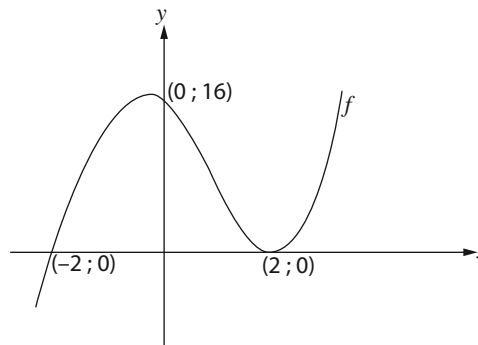
Use $y = a(x - x_1)(x - x_2)(x - x_3)$ Where x_1, x_2 and x_3 are x-intercepts

This is another form of the general formula of a cubic graph. This form is normally used when finding the equation of a cubic graph as opposed to sketching the graph.

The x-intercepts are $x = -2$ and $x = 2$, but at $x = 2$ the graph does not cut the axes, so here we have two x-intercepts that are the same. Making this substitution in the general form, we have:

$y = a(x + 2)(x - 2)(x - 2)$

We are also given the point (0; 16) which is the y-intercept. We can substitute this into out general form to find the value of a:

$16 = a(0 + 2)(0 - 2)(0 - 2)$

$16 = a(2)(-2)(-2)$

$16 = 8a$

$\therefore a = 2$

$y = 2(x + 2)(x - 2)^2$ Substitute $a = 2$ and multiply out.

$y = (2x + 4)(x^2 - 4x + 4)$

$y = 2x^3 - 8x^2 + 8x + 4x^2 - 16x + 16$

$y = 2x^3 - 4x^2 - 8x + 16$

Example 3 (Point of Inflection).

Example

Find the points of inflection and stationary points in $y = x^3 + x^2 + x$.

If we were to look at the function $y = x^3 + x^2 + x$ and try to solve for the stationary points of this function we would get:

$\frac{dy}{dx} = 3x^2 + 2x + 1$

$3x^2 + 2x + 1 = 0$ (solve for the stationary points)

This cannot be factorised easily, so we use the quadratic formula:

$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

$x = -2 \pm \frac{\sqrt{(-2)^2 - 4(3)(1)}}{2(3)}$

$x = -2 \pm \frac{\sqrt{4 - 12}}{6}$

Now we have a problem, there is no solution for x as we have $\sqrt{-8}$ in the quadratic equation. Therefore we can say that there are no stationary points on this curve. However we can find the point of inflection by setting the second derivative equal to zero.

point of inflection $\left(-\frac{1}{3}; -\frac{7}{27}\right)$
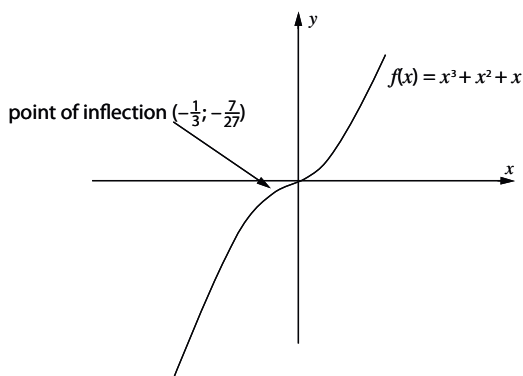
$f(x) = x^3 + x^2 + x$

Activity

Activity 5

1.   The figure represents the graphs of the functions $f(x) = 4x + 2$ and $g(x) = x^3 + x^2 - x - 1$
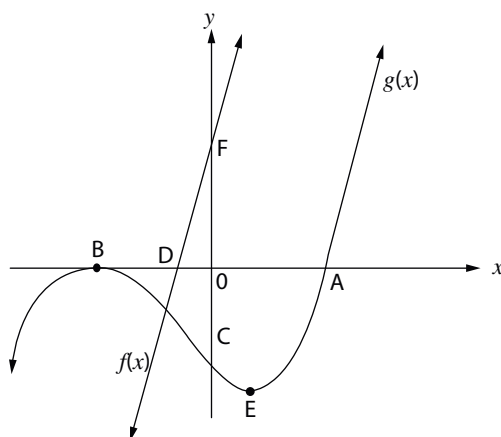
Determine:

1.1   The lengths of DA, OB, and CF.

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1.2   The coordinates of the turning point E.

....

1.3   The value(s) of x for which the gradients of $f(x)$ and $g(x)$ are equal.

....

2. $f(x) = x^3 - 6x^2 + 11x - 6$

2.1   Use a calculator to find the coordinates of the stationary points of $f(x)$, correct to one decimal digit.

....

2.2   Use the remainder theorem to find the roots of $f(x) = 0$

....

2.3   Sketch the curve of $f(x)$, showing all the important points clearly.
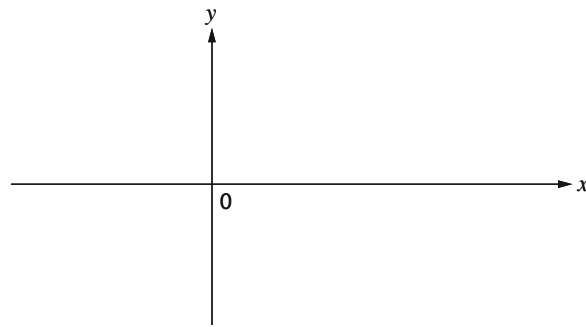
2.4   Determine the gradient of $f(x)$ at each of the x-intercepts.

....

3.   Find an equation for $f(x)$ using the sketch:

LIBERTY LIFE
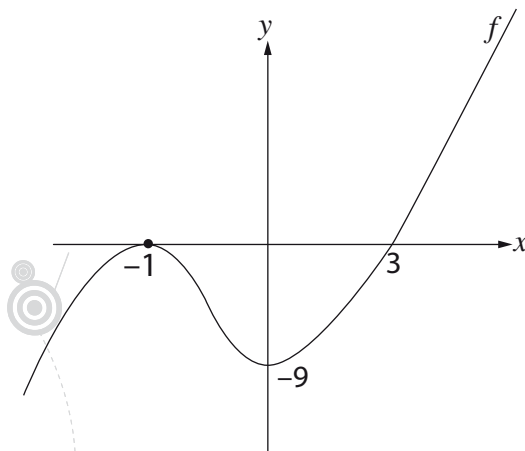
4. Calculate all the stationary points and points of Inflection in the curve $y = 3x^3 + 2x^2 + x$

....

....

....

Maxima and Minima in Calculus

Real Life Application

There are many situations in life where we need to find the maximum or minimum of some quantity. Often businesses need to find ways to minimise costs and maximise productivity. This is where maxima and minima come in very useful.
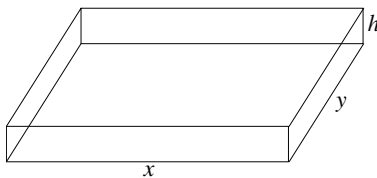
First, let's note some basics that will be useful: often we will get asked to maximise or minimise a certain area, volume, or cost function etc.
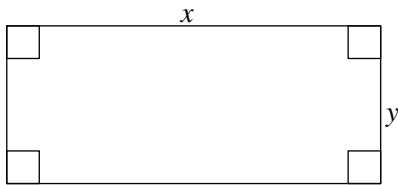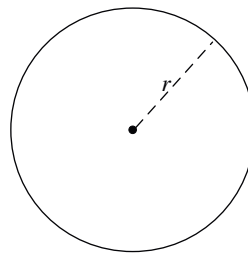
Rectangle

Perimeter $= 2x + 2y$

Area $= xy$

Circle

Perimeter (circumference) $= 2\pi r$

Area $= \pi r^2$

Box

Volume $= xyh$

Surface Area $= 2xy + 2xh + 2yh$....(closed)

Surface Area $= xy + 2xh + 2yh$......(open)
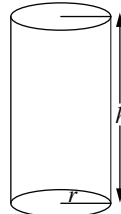
Cylinder

Volume $= \pi r^2 h$

Surface Area $= 2\pi rh + 2\pi r^2$....(closed)

Surface Area $= 2\pi rh + \pi r^2$…(open)

Background

In order to find the maximum or minimum of a function, the derivative of the function must be found and made to equal zero. This is similar to the cubic graphs section, where if we wanted to find the local maximum or minimum we did the same thing. The difference here is that instead of finding a local maximum or minimum of a cubic graph, we may be finding the maximum or minimum volume, area, cost, profit etc.

Example 1

A factory has x employees and makes a profit of P rand per week. The relationship between the profit and number of employees is expressed in the formula $P(x) = -2x^3 + 600x + 1000$, where x is the number of employees

Calculate:

1. The number of employees needed for the factory to make a maximum profit.

   First we need to find the derivative of $P(x)$

   $P'(x) = -6x^2 + 600$

   Next, to find the maximum profit, we need to make $P'(x) = 0$ and then solve for x (the number of employees needed to achieve this profit).

   $P'(x) = 0$

   $\therefore 0 = -6x^2 + 600$

   $6x^2 = 600$

   $x^2 = 100$

   $x = \pm10$ Remember that $(10)^2 = (-10)^2$, so we must include both the positive and negative answers

   But it is not possible to have a negative number of people!

   $\therefore x = 10$

2. The maximum profit

   Now, all we have to do is substitute $x = 10$ back into the formula for $P(x)$

   $P(10) = -2(10)^3 + 600(10) + 1000$

   $P(10) = -2(1000) + 6000 + 1000$

   $P(10) = -2000 + 7000$

   $P(10) = 5000$

   $\therefore$ Max Profit is R5000

Example 2

You have bought 16m of fencing and would like to use it to enclose a field. If length $= h$ and width $= x$, find h in terms of x. What is the maximum area possible for the field?
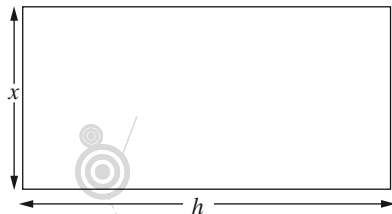
Perimeter $= (2 \times \text{length}) + (2 \times \text{breadth})$ so we can use the 16 m to get a first equation.

$16 = 2h + 2x$

$8 = h + x$

$h = 8 - x$ Now that we have expressed h in terms of x, we can get an equation for the area that does not contain h. When we form the equation that must be differentiated we must always express it in terms of one unknown (usually x).

Area = length × breadth   (this is knowledge you have from grade 9 and 10)

$\quad = xh$ but we know $h = 8 - x$

$\quad = x(8 - x)$

$\quad = 8x - x^2$

For maximum area, we need to find the derivative of the area, then make $\frac{dArea}{dx} = 0$, and solve for x:

$\frac{dArea}{dx} = 8 - 2x$ Now, make this = 0

$0 = 8 - 2x$

$2x = 8$

$x = 4$

Therefore, for the maximum area the width of the field must be 4m.

Now substitute this into the equation for the length: $h = 8 - x$

$\therefore h = 4$

For the maximum possible area of the field, the field must be a square with both a length and a width of 4 m, ie the area = 16 m^2.

Activity 6

1.   The sum of one number and three times another number is 12. Find the two numbers if the sum of their squares is a minimum, **and** give this minimum value.

2.   A rectangular box without a lid is made of card board of negligible thickness. The sides of the base are $2x$ cm by $3x$ cm, and the height is h cm.

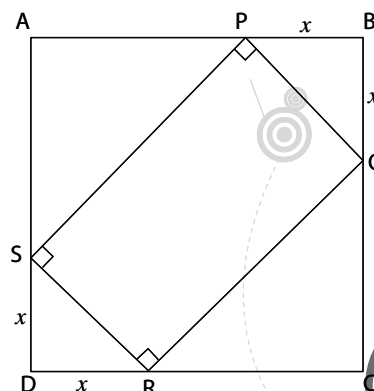2.1   If the total external surface area is 200 cm^2, show that $h = \frac{20}{x} - \frac{3x}{5}$

2.2   Find the dimensions of the box for which the volume is a maximum

3.   ABCD is a square of side 30 cm. PQRS is a rectangle drawn within the square so that BP = BQ = DS = DR = x cm.

3.1   Prove that the area (A) of PQRS is given by A $= 60x - 2x^2$ cm^2

3.2   What is the maximum possible area of PQRS?

Rates of Change

Real Life Application

In any applications that rely on the speed of cars, planes, or even the speed a ball travels through the air, rates of change can be used to calculate the velocity (rate of change of displacement with time) and acceleration (rate of change of velocity with time) of a moving object. So next time you're watching the F1, motorbike racing, or even your favourite sportsman or woman in action, you're watching calculus in motion.

> Velocity/speed = the derivative of displacement/distance with respect to time
>
> Acceleration = the derivative of velocity/speed with respect to time

Example

Example 1

A stone is dropped from the top of a tower. The distance, S metres, which it has fallen after t seconds is given by $S = 5t^2$. Calculate the average speed of the stone between $t = 2$ and $t = 4$ seconds.

Solution

Solution:

As the question asked for average speed, we need to find the average gradient of the curve defined by $S = 5t^2$. So first we need to find the value of S at the two points $t = 2$ and $t = 4$:

$S = 5(2)^2$	$S = 5(4)^2$
$= 5(4)$	$= 5(16)$
$= 20$	$= 80$

$$\text{Average speed} = \frac{S(4) - S(2)}{4 - 2} \text{ m/s}$$
$$= \frac{80 - 20}{4 - 2} \text{ m/s}$$
$$= \frac{60}{2}$$
$$= 30 \text{ m/s}$$

Example

Example 2

An object is projected vertically upwards from the ground and its movement is described by $S(t) = 112t - 16t^2$, where S = distance from the starting point in metres and t = time in seconds.

1.   Determine its initial velocity

The initial velocity of the object is its velocity at $t = 0$. This is also the rate of change of distance at $t = 0$. Therefore we need to find the formula for the derivative of $S(t)$ and evaluate it at $t = 0$:

$v(t) = \frac{dS}{dt} = 112 - 32t$. Remember velocity is the derivative of distance/displacement.

$v(0) = 112 - 32(0)$ substitute $= 0$

$= 112$ m/s   this is the initial velocity

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2.   Determine its velocity after 3 seconds

Now we substitute $t = 3$ into the formula we have found for the velocity

$v(3) = 112 - 32(3)$

$= 112 - 96$

$= 16$ m/s   this is the velocity after 3 seconds

3.   Find the maximum height the object reaches

This will be the maximum displacement of the object which, from maxima and minima, we can find by putting $s'(t) = 0$ and solving for t:

$S'(t) = 112 - 32t$

$112 - 32t = 0$ put equal to zero to find t at maximums

$112 = 32t$

$t = \frac{112}{32} = \frac{7}{2} = 3.5$ seconds

$S\left(\frac{7}{2}\right) = 112\left(\frac{7}{2}\right) - 16\left(\frac{7}{2}\right)^2$ substitute $t = 3.5$ into $S(t)$ to find maximum height

$= 392 - 196$

$= 196$

It reaches a height of 196m

4.   When will it be more than 96m above the ground?

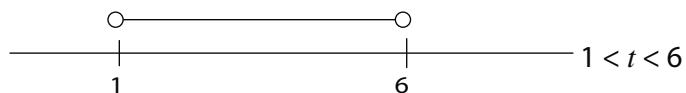Now we need to create an inequality for $S(t) > 96$m and solve for t:

$112t - 16t^2 > 96$

$0 > 16t^2 - 112t + 96$ move everything to the same side of the inequality sign

$t^2 - 7t + 6 < 0$ factorise and simplify

$(t - 6)(t - 1) < 0$

$1 < t < 6$

It is above 96 m from 1 to 6 seconds.

5.   What is the acceleration of the object?

Acceleration is the change in velocity with respect to time. So we need to find the derivative of the velocity:

$v(t) = 112 - 32t$

$a(t) = v'(t)$

$a(t) = -32$ m·s⁻²

Activity 7

1.   A large tank is being filled with water through certain pipes. The volume of water in the tank at any given time is given by $V(t) = 5 + 10t - t^2$, where the volume is measured in cubic metres and the time in minutes.

1.1   Find the rate at which the volume is increasing when $t = 1$ minute

1.2   At what time does the water start running out of the tank?

2.   A cricket ball is thrown vertically up into the air. After x seconds, its height is y metres where $y = 50x - 5x^2$. Determine:

2.1   the velocity of the ball after 3 seconds

2.2   the maximum height reached by the ball.

2.3   the acceleration of the ball.

2.4   the total distance travelled by the ball when it returns to the ground.

Activity

Activity 8

1.   Differentiate the following with respect to x:

1.1 $(x - \frac{1}{x})(x + \frac{1}{x})$

1.2 $\dfrac{x^3 - 2x^2 - x + 2}{x - 2}$

1.3 $3 - 4x^3$

1.4 $\sqrt{x} + 3x - \dfrac{4}{x^2}$

2.   Determine: $\lim\limits_{x \to 2} \dfrac{x(x^2 - x - 2)}{x - 2}$

3.   If $xy = 2 - x\sqrt{x}$, determine $\dfrac{dy}{dx}$. Give the answer with positive exponents.

4.   Evaluate: $\lim\limits_{x \to 2} \dfrac{x^3 - 8}{x^2 - 4}$

5.   Determine $f\,'(x)$ from first principles if $f(x) = -3\,x^2$

6.   Find the average gradient of $g(x)$ between $x = 1$ and $x = 3$ if $g(x) = 2x^2 - 2$

7.   Determine $\dfrac{dy}{dx}$ if:

7.1 $y = (x^3 - 1)^2$

7.2 $y = \dfrac{x^3 + \sqrt{x^3}}{x}$

7.3 $y = (x - x^{-1})^2$

7.4 $8 = xy$

7.5 $y = \dfrac{1 - 2x + \sqrt{x}}{x^2}$

8.   If $f(x) = -\dfrac{x^2}{2} + \dfrac{2}{x^2} + x$, find $f\,'(x)$ and hence evaluate $f\,'(-2)$

1.   The figure shows the curve given by $f(x) = ax^3 + 1$. Point P (2 ; 9) lies on the curve of $f(x)$.
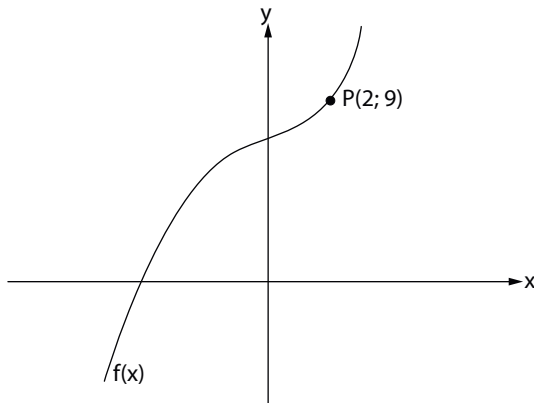
1.1   Determine the value of a.

1.2   Determine the equation of the tangent at point P.

1.3   Determine the average gradient of the curve between $x = 2$ and $x = 5$

2.   Given: $f'(x) = 0$ at $x = -3$ and $x = -1$ only and that $f(-5) = 0$; $f(-2) = 0$; $f(1) = 0$; $f(0) = -1$. Draw a rough sketch of a graph of $f(x)$ assuming it is a cubic polynomial.

3. $f(x) = -x^3 + bx^2 + cx + d$.

      A and B are turning points with B(6; 0)
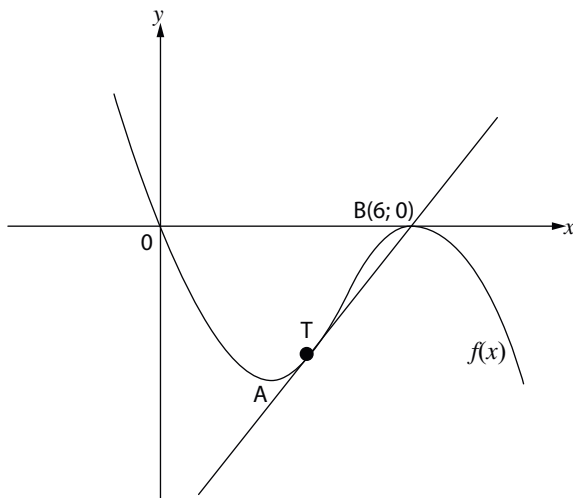
      Find:

3.1   the values of b, c and d.

3.2   the coordinates of A

3.3   T has an x-coordinates of 3, find the y-coordinate of T

3.4   Equation of the tangent to the curve at T

4.1   Show that the gradient of the tangent at (2; 4) to the curve $y = -1 + 3x - \dfrac{x^2}{4}$ is double that of the gradient at the point (4; 7)

4.2   Determine the co-ordinates of the point on the curve at which the gradient is -1.

Solutions to Activities

Activity 1

1. 27

2. $\lim\limits_{x \to -1} (x + 1) = 0$

3. $\dfrac{5}{4}$

Activity 2

1.1 gradient $= \dfrac{f(6) - 8(0)}{6 - 0} = \dfrac{-27 - 9}{6 - 0} = -6$

1.2 $x = 3: f'(n) = \lim\limits_{h \to 0} \dfrac{f(3 + h) - f(3)}{h}$

$\lim\limits_{h \to 0} \dfrac{-(3 + h)2 + 9 - (-3)^2 + 9)}{h}$

$\lim\limits_{h \to 0} \dfrac{-9 - 6h - h^2 + 9 + 9 - 9}{h}$

$\lim\limits_{h \to 0} \dfrac{h(-6 - h)}{h}$

$-6 - 0 \qquad = -6$

2.1 $f'(x) = \lim\limits_{h \to 0} \dfrac{(x + h)^2 + 2 - (x^2 + 2)}{h}$

$\lim\limits_{h \to 0} \dfrac{x^2 + 2xh + h^2 + 2 - x^2 - 2}{h}$

$\lim\limits_{h \to 0} \dfrac{2xh + h^2}{h}$

$\lim\limits_{h \to 0} \dfrac{h(2x + h)}{h}$

$\lim\limits_{h \to 0} (2x + h) \qquad = 2x$

2.2 $f'(x) = \lim\limits_{h \to 0} \dfrac{3(x + h)^2 + 3(x + h) - (3x^2 + 3x)}{h}$

$\lim\limits_{h \to 0} \dfrac{3x^2 + 6xh + 3h^2 + 3x + 3h - 3x^2 - 3x}{h}$

$\lim\limits_{h \to 0} \dfrac{h(6x + 3h + 3)}{h}$

$\lim\limits_{h \to 0} (6x + 3h + 3) \qquad = 6x + 3$

Activty 3

1. $f'(x) = \dfrac{1}{2}x - \dfrac{1}{2} - \dfrac{1}{2}x - \dfrac{3}{2}x$

2. $\therefore y = 2x^3(4x^2 - 12x + 9)$

$\therefore 8x^5 - 24x^4 + 18x^3$

$\dfrac{dy}{dx} = 40x^4 - 96x^3 + 54x^2$

$y = x - 2 + 2x^{-1}$

3. $\dfrac{dy}{dx} = 1 - 2x^{-2}$

Activity 4

1. $f(x) = x^2 - 2x + 1 \qquad y$ into: $(0; 1)$

$f'(x) = 2x - 2$

$\therefore$ gradient of tangent $x: f'(0) = 2(0) - 2 = -2$

$\therefore$ tangent: $y = -2c$ (sub$(0; 1)$)

$\therefore y = -2n + 1$

2. $f(x) = -x^2 + x + 2$

$$f'(x) = -2x + 1$$

at $x = 1$: $\quad f'(1) = -2(1) + 1 = -1$

$\therefore$ m of tangent $= -1$

$\therefore$ tangent: $y = -x + c$ $\qquad$ sub $(1 ; 2)$

$$2 = -1 + c$$

$$\therefore 3 = c$$

$\therefore$ tangent: $y = -x + 3$

3. $\qquad y = 11x + 44$ is a tangent to $f(x) = -2x^2 - 5x + 12$

$\therefore$ m = 11 and gradient: $\quad f'(x) = -4x - 5$

$$\therefore -4x - 5 = 11$$

$$-4x = 16$$

$$\therefore x = -4$$

to find y, sub $x = -4$ into either equation:

$y = 11(-4) + 44 \qquad$ or $\qquad f(-4) = -2(-4)^2 - 5(-4) + 12$

$\quad = 0 \qquad\qquad\qquad\qquad\qquad = 0$

$\therefore$ point of contact: (-4 ; 0)

4. $\quad f(x) = \frac{3}{x} = 3x^{-1}$

$f'(x) = -3x^{-2} = \frac{-3}{x^2}$

at $x = -1$: $\quad f'(-1) = \frac{-3}{(-1)^2} = \frac{-3}{1} = -3 \qquad \therefore$ m of tangent $= -3$

to find the y-value: $\quad f(-1) = \frac{3}{-1} = -3$

$\therefore$ point of contact: $(-1 ; -3)$

$\therefore$ tangent: $\quad y = -3x + c \qquad\qquad$ sub $(-1 ; -3)$

$$\therefore -3 = -3(-1) + c$$

$$-6 = c$$

$\therefore$ tangent: $\quad y = -3x - 6$

Activity 5

1. $\quad$ Finding x-intercepts of $g(x)$: $(y = 0)$

$\therefore x^3 + x^2 - x - 1 = 0$

$(x + 1)(x - 1)(x + 1) = 0$

$\therefore x = -1$ or $x = 1$

$\therefore$ B(-1; 0) and A(1; 0)

y-intercept of $g(x)$: $-1 \qquad \therefore$ C(0; -1)

Finding x-intercept of $f(x)$: $(y = 0)$

$4x + 2 = 0$

$\therefore x = -\frac{1}{2}$

$\therefore D\left(-\frac{1}{2}; 0\right)$

1.1 $DA = 1 + \frac{1}{2} = \frac{3}{2}$

  $OB = 1$

  $CF = 1 + 2 = 3$

1.2 $g'(x) = 3x^2 + 2x - 1 = 0$

  $(3x - 1)(x + 1) = 0$

  $\therefore x = \frac{1}{3}$ or $x = 1$

  At E, $x = \frac{1}{3}$ $\therefore y = g\left(\frac{1}{3}\right) = -\frac{32}{27}$

  $\therefore E\left(\frac{1}{3}; -\frac{32}{27}\right)$

1.3 Gradient of $f(x) = 4$ (from $m = 4$)

  $\therefore g'(x) = 3x^2 + 2x - 1 = 4$

  $\therefore 3x^2 + 2x - 5 = 0$

  $(3x + 5)(x - 1) = 0$

  $x = -\frac{5}{3}$ or $x = 1$ (values of x for which the gradient of $f(x)$ and $g(x)$ are equal.)

2.1 $f'(x) = 3x^2 - 12x + 11 = 0$

  $x = 2{,}577\ldots$ or $x = 1{,}422\ldots$

  $\therefore x = 2{,}6$ or $x = 1{,}4$

  $\therefore f(2{,}6) = -0{,}4$ $f(1{,}4) = (0{,}4)$

  $(2{,}6; -0{,}4)$ $(1{,}4; 0{,}4)$

2.2 $(x - 1)(x - 2)(x - 3) = 0$ (x-intercepts)

  $\therefore x = 1$ or $x = 2$ or $x = 3$

2.3
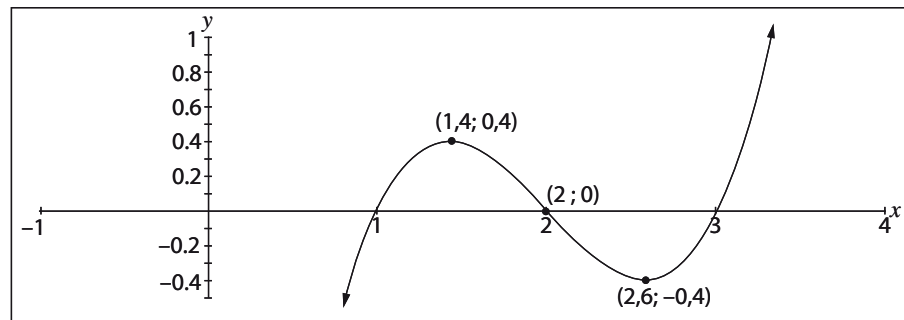
2.4 $x = 1; m = 2$

  $x = 2; m = -1$

  $x = 3; m = 2$

2.5 $\therefore y = -\frac{1}{2}x + c$ (sub$(1; 0)$) Note: a normal is a line $\perp$ to the tangent

             m of tangent $= 2 \therefore m$ of normal $= -\frac{1}{2}$

  $\therefore y = -\frac{1}{2}x + \frac{1}{2}$

3. $y = a(x + 1)(x + 1)(x - 3)$ sub$(0; -9)$

  $\therefore -9 = a(1)(1)(-3)$

  $-9 = -3a$ $\therefore a = 3$

  $\therefore y = 3(x + 1)(x + 1)(x - 3)$

  $\therefore y = 3x^3 - 3x^2 - 15x - 9$

$$y = 3x^3 - 3x^2 - 15x - 9$$

4. No stationary points since derivative $9x^2 + 4x + 1 = 0$ has no solution. However the second derivative equal to zero **does** have an answer: $18x + 4 = 0$ so $x = -\frac{2}{9}; y = -\frac{38}{243}.$

Activity 6

1. The numbers are $\frac{18}{5}$ and $\frac{6}{5}$; the minimum is $14\frac{2}{5}$

2.2 Dimensions are $\frac{20}{3}$ cm; 4cm; 10cm

3.1 Using Pythag: SR $= \sqrt{2}\, x$

Note: RC = QC = $30 - x$

$\therefore$ QR$^2 = (30 - x)^2 + (30 - x)^2$

$2(30 - x)^2$

$\therefore$ QR $= \sqrt{2}\,(30 - x)$

$\therefore$ Area PQRS $= (\sqrt{2}\, n)(\sqrt{2}(30 - x))$

$\therefore$ A $= 2(30x - x^2)$

$\therefore$ A $= 60x - 2n^2$ cm^2

3.2 max Area: A$' = 60 - 4x = 0$

$\therefore x = 15$

$\therefore$ max area A(15) $= 60(15) - 2(5)^2$

450 cm^2

Activity 7

1.1 Rate of increase: V$'$(t) $= 10 - 2t$

at t $= 1$: V$'$(1) $= 8$m^3/minute

1.2 max volume: V$'$(t) $= 10 - 2t = 0$

$\therefore t = 5$ min

2.1 velocity $= \frac{dy}{dx} \therefore \frac{dy}{dx} = 50 - 10x$

$x = 3$: velocity $= 50 - 30 = 20$ m/s

2.2 max height: $\frac{dy}{dx} = 50 - 10x = 0$

$\therefore x = 5$

$\therefore$ Max height at $x = 5$: $y = 50(5) - 5(5)^2 = 125$m

2.3 Acceleration $= \frac{d^2y}{dx^2} = -10$ m/s^2

2.4 Double the maximum $= 250$ m

Activity 8

1.1 $x^2 - \frac{1}{x^2} = x^2 - x^{-2}$ $\quad \therefore$ derivative $= 2x + \frac{2}{x^3}$

1.2 $\frac{(x-2)(x^2 - 1)}{(x-2)} = x^2 - 1 \therefore$ derivative $= 2x$

1.3 derivative $= -12x^2$

1.4 $x^{\frac{1}{2}} + 3x - 4x^{-2} \therefore$ derivative $= \frac{1}{2}x^{-\frac{1}{2}} + 3 + 8x^{-3}$

$$= \frac{1}{2x^{\frac{1}{2}}} + 3 + \frac{8}{x^3}$$

2. $\displaystyle\lim_{x \to 2} \frac{x(x-2)(x + 1)}{(x-2)} = \lim_{x \to 2} x(x + 1) = 2(2 + 1) = 6$

3. $xy = 2 - x\sqrt{x}$ Make y the subject of the formula.

$y = \dfrac{2}{x} - \dfrac{x\sqrt{x}}{x}$

$\therefore y = 2x^{-1} - x^{\frac{1}{2}}$

$\therefore \dfrac{dy}{dx} = \dfrac{2}{x^2} - \dfrac{1}{2x^{\frac{1}{2}}}$

4. $\displaystyle\lim_{x \to 2} \dfrac{(x-2)(x^2 + 2x + 4)}{(x-2)(x + 2)} = \dfrac{2^2 + 2(2) + 4}{2 + 2} = 3$

5. $f'(x) = \displaystyle\lim_{h \to 0} \dfrac{f(x + h) - f(x)}{h} = \lim_{h \to 0} \dfrac{-3(x + h)^2 - (-3x^2)}{h}$

$= \displaystyle\lim_{h \to 0} \dfrac{-3x^2 - 6xh - 3h^2 + 3x^2}{h}$

$= -6x - 3(0) = -6x$

6. $m = \dfrac{g(3) - g(1)}{3 - 1} = \dfrac{16 - 0}{2} = 8$

7.1 $y = x^6 - 2x^3 + 1$ $\therefore \dfrac{dy}{dx} = 6x^5 - 6x^2$

7.2 $y = x^2 + \dfrac{x^{\frac{3}{2}}}{x} = x^2 + x^{\frac{1}{2}}$ $\therefore \dfrac{dy}{dx} = 2x + \dfrac{1}{2\,x^{\frac{1}{2}}}$

7.3 $y = x^2 - 2 + x^{-2}$ $\therefore \dfrac{dy}{dx} = 2x - \dfrac{2}{x^3}$

7.4 $y = \dfrac{8}{x} = 8x^{-1}$ $\therefore \dfrac{dy}{dx} = \dfrac{-8}{x^2}$

7.5 $y = \dfrac{1}{x^2} - \dfrac{2}{x} + \dfrac{x^{\frac{1}{2}}}{x^2} = x^{-2} - 2x^{-1} + x^{\frac{-3}{2}}$

$\therefore \dfrac{dy}{dx} = \dfrac{-2}{x^3} + \dfrac{2}{x^2} - \dfrac{3}{2x^{\frac{5}{2}}}$

8. $f(x) = -\dfrac{1}{2}x^2 + 2x^{-2} + x$ $\therefore f'(x) = -x - \dfrac{4}{x^3} + 1$

$\therefore f'(-2) = 2\dfrac{-4}{-8} + 1$

$= \dfrac{7}{2}$

9. $f'(x) = 7x_2^5 + 18x^2 + 10x_2^3$

Activity 9

1.1 sub $(2; 9) : 9 = a(2)^3 + 1$

$\therefore a = 1$

1.2 gradient at $x = 2 : f'(x) = 3x^2$

$\therefore f'(2) = 12$

$\therefore$ tangent: $y = 12x + c$ sub $(2; 9)$

$\therefore c = -15$ and $y = 12x - 15$

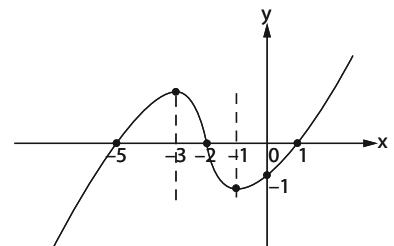1.3 $m = \dfrac{f(5) - f(2)}{5 - 2} = \dfrac{126 - 9}{5 - 2} = 39$

2. Note: $f'(-3) = 0$ and $f'(-1) = 0$

$\therefore$ these are stationary points.

$-f(-5) = 0; f(-2) = 0; f(1) = 0$

$\therefore$ these are the x-intercepts

$f(0) = -1$ ie. -1 is the y-intercept value

3. $d = 0$ (graph passes through the origin)

3.1   to find b and c (2 unknowns) we solve simultaneously.

        from B $f(6) = 0$ $\therefore -(6)^3 + 6(6)^2 + c(6) + 0 = 0$

                                   $6b + c = 36$

                                 $\therefore c = 36 - 6b$

since B is a stationary point $f'(6) = 0 : f'(x) = -3x^2 + 2bx + c = 0$

                  $f'(6) = -3(6)^2 + 2b(6) + c = 0$

        (from eqn 1: $c = 36 - 6b$) $\therefore -108 + 12b + c = 0$

                                 $\therefore -108 + 12b + 36 - 6b = 0$

             $6b = 72$

             $b = 12$

             $\therefore c = -36$

3.2   eqn: $y = -x^3 + 12x^2 - 36x$

        $\therefore \dfrac{dy}{dx} = -3x^2 + 24x - 36$

        at A (stationary point) : $-3x^2 + 24x - 36 = 0$

        $x^2 - 8x + 12 = 0$

        $(x - 6)(x - 2) = 0$

        $\therefore A \to x = 2$

        $f(2) = -32$ $\therefore A(2; -32)$

3.3 $f(3) = -27$ $\therefore y_T = -27$

3.4 m of tangent: $f'(3) = -3(3)^2 + 24(3) - 36$

        $= 9$

        $\therefore y = 9x + c$ sub T(3; -27)

        $\therefore C = -54$ $\therefore y = 9x - 54$

4. $\dfrac{dy}{dx} = 3 - \dfrac{1}{2}x$

        gradient at $x = 2$: $3 - \dfrac{1}{2}(2) = 2$

        gradient at $x = 4$: $3 - \dfrac{1}{2}(4) = 1$

        $\therefore$ gradient at (2; 4) is double that at point (4; 7)

4.1   (8; 7)

6.1   OP = 2 units; OQ = 1 unit; OR = 6 units; ON = 12 units

6.2   S(4; -36)

6.3   OT = 30 units; TN = 3 units

6.4 $y = -11x + 3$

6.5   Z(-1; 14)

7. $p = \dfrac{9}{4}$

8.   (1; s7)